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INVERSE STEFAN PROBLEMS

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We consider statements and a method for obtaining stationary solutions of boundary-value and coefficient inverse problems for a quasilinear Stefan problem.

1°. Stefan problems in direct statement are boundary-value problems for parabolic equations in the regions with free boundaries on which the Stefan conditions of material or energy balance are imposed. In a thermophysical interpretation, these problems consist in determining the temperature distribution $u(x, t)$ and the fronts of phase transitions $\xi(t)$ (one or several) under the assumption that all coefficients of the equation and the Stefan condition and all initial and boundary functions are specified (see, for example, [1, 2]). Each direct Stefan problem can be associated with a set of inverse problems, if, apart from the functions $u(x, t)$ and $\xi(t)$, some functions regarded as given in the direct statement must be determined from certain additional information. In accordance with the sought characteristic of the mathematical model, inverse Stefan problems can be divided into boundary-value, coefficient, and retrospective ones. Just as many inverse problems of mathematical physics, inverse Stefan problems are incorrect [3], i.e., requirements for existence, uniqueness, and stationarity of a solution may be not fulfilled for them. The development of the theory and methods for solving this class of incorrect problems is prompted by the requirements of mathematical modeling of complex nonlinear processes with phase transitions in thermophysics and the mechanics of a continuous medium in connection with problems of improving technologies and creating new methods of treating materials and up-to-date engineering specimens.

The range of practical applicability of inverse Stefan problems is extensive. Specific examples of these problems are presented in [4, 5]. Inverse Stefan problems for quasilinear parabolic equations are particularly topical because their solution, as a computerized calculational experiment, is in some cases practically the only means of studying high-temperature problems with phase transitions (in which the temperature dependence of the thermophysical characteristics should be taken into account). Quasilinear inverse Stefan problems are investigated in [6-8] and other works.

2°. We present some statements of boundary-value and coefficient inverse problems for one of the common versions of the quasilinear Stefan problem, viz., a two-phase problem with a single inner front:

$$c(x, t, u)\, u_t - Lu = 0, \quad (x, t) \in Q_1 = \{0 < x < \xi(t), \quad 0 < t \le T\},$$
$$(x, t) \in Q_2 = \{\xi(t) < x < l, \quad 0 < t \le T\}, \tag{1}$$

$$u|_{x=0} = v(t), \quad 0 < t \le T, \tag{2}$$

$$a(x, t, u)\, u_x + e(t)\, u|_{x=0} = p(t), \quad 0 < t \le T, \tag{3}$$

$$u|_{t=0} = \varphi(x), \quad 0 \le x \le l, \tag{4}$$

$$u|_{x=\xi(t)} = u^*(t), \quad 0 < t \le T, \tag{5}$$

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$$\gamma(x, t, u)|_{x=\xi(t)}\ \xi_t = [a(x, t, u)u_x]|_{x=\xi(t)} + \chi(x, t, u)|_{x=\xi(t)}, \quad 0 < t \le T, \tag{6}$$

$$\xi|_{t=0} = \eta_0, \tag{7}$$

where $Lu \equiv (a(x, t, u)u_x)_x - b(x, t, u)u_x - d(x, t)u + f(x, t, u)$ is a uniformly elliptic operator and $[w]_{x=\xi(t)} = w|_{x=\xi(t)+0} - w|_{x=\xi(t)-0}$. In the direct statement, $a \ge a_{\min} > 0$, b, $c \ge c_{\min} > 0$, d, f, v, e, p, φ, u^*, $\gamma \ge \gamma_{\min} > 0$, χ are known functions of their arguments, and $\eta_0 = \text{const} > 0$.

Statement I. We assume that the boundary conditions at $x = 0$ are not known (i.e., the function $v(t)$ in Eq. (2) is not known) but at the other boundary $x = l$, aside from condition (3), additional information on the solution of the direct Stefan problem (1)-(7) is given:

$$u|_{x=l} = g(t), \quad 0 \le t \le T, \tag{8}$$

where $g(t)$ is a function known for $0 \le t \le T$. Then a boundary-value inverse Stefan problem arises, viz., to find the functions $u(x, t)$ in the region $Q = Q_1 \cup Q_2$, $\xi(t)$, and $v(t)$ for $0 \le t \le T$ that satisfy conditions (1)-(8), in which the input data a, b, c, d, f, e, p, u^*, γ, χ, φ, g, and η_0 are assumed to be given.

This problem may be considered as that of extending the solution of the quasilinear parabolic equation (1) from the boundary $x = l$, where the Cauchy conditions (3) and (8) are specified, into the region Q, in which phase transitions occur. Thus, it may be included among noncharacteristic Cauchy problems for parabolic equations, but a substantial complication is posed by an unknown phase transition front that moves with time inside the region and divides it into two parts.

In a thermophysical interpretation, inverse problem (1)-(8) consists in determining the thermal conditions at the boundary $x = 0$ from the given temperature and heat flux at the boundary $x = l$, for example, when it is impossible to measure the temperature on part of the surface of the object considered. Additional information can be given not at the boundary $x = l$ but, instead, at the interior point $u|_{x=l_0} = g(t)$, $0 < l_0 < l$, if, for example, the temperature can be measured inside the body.

We now represent inverse problem (1)-(8) as the operator equation

$$Sv = g, \quad v \in V, \quad g \in G, \quad V \subset L_2[0, T], \quad G \subset L_2[0, T], \tag{9}$$

where S is a nonlinear operator that puts the spur of the solution $u|_{x=l}$ of the direct Stefan problem (1)-(7) in correspondence with each element $v \in V$. An exact solution of Eq. (9) is an element $v^0 \in V$ for which the solution spur at $x = l$ coincides with the given element $g \in G$.

The requirements of classical solvability of the direct Stefan problem (1)-(7) permit selection of "natural" functional spaces for the input data and the solution of the inverse problem that provide determination of the operator S and uniqueness of the exact solution v^0 (if any) [8]. Based on these requirements, we refine Eq. (9), choosing

$$V = \left\{ v(t) \in W_2^2[0, T], \quad c(0, 0, \varphi)v_t - L\varphi|_{\substack{x=0\\t=0}} = 0 \right\}, \tag{10}$$

$$G = \left\{ w(t) \in C^{1+\lambda/2}[0, T], \quad c(l, 0, \varphi)w_t - L\varphi|_{\substack{x=l\\t=0}} = 0 \right\}. \tag{11}$$

N o t e 1. If the boundary conditions at $x = 0$ are sought in the form

$$q(t) = a(x, t, u)u_x|_{x=0}, \quad 0 \le t \le T, \tag{12}$$

the operator representation of this inverse problem is the following:

$$Sq = g, \quad q \in \Theta, \quad g \in G,$$

where G is defined in Eq. (11), the set Θ is selected from the conditions of classical solvability of the direct Stefan problem (1), (12), and (3)-(7), and S: $\Theta \to G$ is a nonlinear operator that sets up a correspondence between each element $q \in \Theta$ and the solution spur $u|_{x=l}$ of the corresponding direct problem [8].

Statement II. Another boundary-value inverse Stefan problem on finding unknown boundary conditions at $x = 0$ arises when additional information on the solution of the direct problem (1)-(7) is given at the final instant of time $t = T$:

$$u|_{t=T} = g(x), \quad 0 \le x \le l, \quad \xi|_{t=T} = \eta, \tag{13}$$

where $g(x)$ is a function known for $x \ge 0$, $\eta \ge 0$ is a known constant, and $T > 0$ is a given instant of time. Here it is assumed that the coefficients of Eq. (1) and the Stefan condition (6), conditions (3) and (5) at the boundary $x = l$ and at the phase transition front, and the initial data (4) and (7) are known functions of their arguments. It is required to define functions $u(x, t)$ in Q, $\xi(t)$, and $v(t)$ for $0 \le t \le T$ that satisfy conditions (1)-(7) and (13).

An inverse Stefan problem of this type models, for example, a controllable thermophysical process with a phase transition and consists in finding the controlling boundary conditions that ensure the desired course of the process. Incorrectness of this boundary-value inverse Stefan problem, apart from the absence of an exact solution with mismatched specification of the input data, manifests itself in the violation of the requirements of solution stationarity and uniqueness. Its operator representation is of the form

$$Sv = z, \quad v \in V, \quad z \in Z, \tag{14}$$

where S is a nonlinear operator that puts each element $v \in V$, $V \subset L_2[0, T]$, in correspondence with a solution $\{u|_{t=T}, \xi|_{t=T}\}$ of the direct Stefan problem (1)-(7) at final time instant of time. An exact solution of Eq. (14) is an element $v^0 \in V$ for which the corresponding solution of problem (1)-(7) at $t = T$ coincides with the element $z \in Z$, where $z = \{g, n\}$, $Z = G \times E$, g is given element of the functional space G, and η is a given number of the set of real numbers E.

The possibility of determining the operator S is provided by the choice of spaces for the input data based on the conditions of classical solvability of the Stefan problem (1)-(7). We assume, in particular, that the set V in Eq. (14) has the form (10), and $G = \{w(x) \in C^{2+\lambda}[0, 1]\}$, $0 < \lambda < 1$.

Statement III. We discuss one more boundary-value inverse problem for the quasilinear Stefan problem (1)-(7), namely, to determine from the given information (13) functions $u(x, t)$ in the region Q, $\xi(t)$, and $u^*(t)$ for $0 \le t \le T$ that satisfy conditions (1)-(7) and (13) under the assumption that the coefficients of Eq. (1) and the Stefan condition (6), the boundary conditions (2) and (3), and the initial data (4) and (7) are known functions of their arguments.

Possible areas where such inverse Stefan problems may arise are mathematical modeling of thermophysical processes with an unknown temperature of a phase transition and also of some diffusion and filtration processes in porous bodies (for example, in studying and exploiting oil and gas deposits).

The corresponding operator representation is

$$Su^* = z, \quad u^* \in \mathscr{U}, \quad z \in Z, \tag{15}$$

where $S: \mathscr{U} \to Z$ is a nonlinear operator that puts each element $u^* \in \mathscr{U}$ in correspondence with a solution $\{u|_{t=T}, \xi|_{t=T}\}$ of the direct Stefan problem (1)-(7) at the final instant of time. An exact solution $u^{*0} \in \mathscr{U}$ of Eq. (15) (determined by analogy with Statement II) may be nonexistent; otherwise it does not possess the properties of stationarity and uniqueness. The corresponding requirements of smoothness and matching of input data make it possible to obtain the operator S for any $u^* \in \mathscr{U}$, which is chosen in the form

$$\mathscr{U} = \left\{u^*(t) \in W_2^2[0, T], \quad c(x, 0, \varphi)\, u_t^* - L\varphi\,\Big|_{\substack{x=\eta_0 \\ t=0}} = 0\right\}.$$

Statement IV. We examine the coefficient inverse Stefan problem: to find functions $u(x, t)$ in the region $Q = Q_1 \cup Q_2$ and $\xi(t)$ for $0 \le t \le T$ and a coefficient $f(x, t, u)$ in the regions $D_1 = Q_1 \times [-M^1, M^1]$, $D_2 = Q_2 \times [-M^2, M^2]$ ($M^k = \text{const} > 0$, $\max\limits_{(x,t)\in \bar{Q}_k} |u| \le M^k$, $k = 1, 2$) that satisfy conditions (1)-(7) and the additional condition (13). Specific examples of this coefficient problem are given in [5, 7] and are concerned with determining an optimal intensity distribution of a laser source that ensures the desired course of a melting process. We represent the problem as the operator equation

$$Sf = z, \quad f \in F, \quad z \in Z, \tag{16}$$

where S is a nonlinear operator setting up a correspondence between each element f of the set F and a solution $\{u|_{t=T}, \xi|_{t=T}\}$ of the direct problem (1)-(7) at $t = T$. An exact solution of Eq. (16) is an element $f^0 \in F$ for which the solution of the direct Stefan problem coincides, at $t = T$, with the element $z \in Z$, $z = \{g, \eta\}$, $Z = G \times E$ (g is a given element of the space G, and η is a given number of the set of real numbers E). The incorrectness of the problem is exhibited by possible absence of the element f^0 (with mismatched input data) and violations of the requirements of uniqueness and stationarity. The possibility of determining the operator S for any $f \in F$ is provided by the corresponding selection of functional spaces for the input data and the solution with allowance for the conditions of classical solvability of the quasilinear Stefan problem (1)-(7).

N o t e 2. Obviously, aside from those considered, other statements of inverse problems are possible for the two-phase Stefan problem (1)-(7), depending on the sought causal characteristic and the way of specifying additional information. There are also corresponding statements for other versions of the quasilinear Stefan problem (uniphase or multiphase), the set of solutions of which comprises a domain of the values of operator S.

3^0. Because of their incorrectness and nonlinearity, a numerical solution of inverse Stefan problems presents significant difficulties and calls for special regularization methods and computational algorithms. Works [6-8] proposed and substantiated a regularization method of variational type that relies on constructing quasisolutions and permits obtaining stationary approximate solutions. We briefly outline the essence of the method using the operator equation (9) as an example.

By a quasisolution of Eq. (9) on the compact set

$$V_R = \left\{ v \in V, \quad \| v \|_{W_2^2[0,T]} \le R \right\}, \quad R = \text{const} > 0,$$

in $C^{1+\lambda/2}[0, T]$ $(0 < \lambda < 1)$ we mean a set

$$V_R^* = \left\{ v_R \in V_R, \quad J_g(v_R) = \inf_{v \in V_R} J_g(v) \right\}, \quad J_g(v) = \| Sv - g \|_{L_2[0,T]}.$$

The correctness of the problem of minimizing the functional $J_g(v)$ on V_R and the possibility of determining the quasisolution V_R^* (nonnullness of V_R^*) result from the fact that $J_g(v)$ is continuous in $C^{1+\lambda/2}[0, T]$ on the compact set V_R.

If the equality $\inf\limits_{v \in V_{\bar{R}}} J_g(v) = 0$ holds for a certain set $V_{\bar{R}}$, then the exact solution v^0 belongs to $V_{\bar{R}}$ and the quasisolution $V_{\bar{R}}^*$ consists of the single element v^0 by virtue of the uniqueness of the exact solution of the inverse Stefan problem in Statement I. Thus, the original problem is reduced to a variational one, for which all correctness conditions are fulfilled. Should $\inf\limits_{v \in V_{\bar{R}}} J_g(v) > 0$, then the following assertion is true for the quasisolutions V_R^* on the compact sets V_R with $\bar{R} \le R \le R^0 = \| v^0 \|_{W_2^2[0,T]}$: any element v_R of V_R^* converges in $W_2^2[0, T]$ to v^0 when $R \to R^0$. In this case, the corresponding solution of the direct Stefan problem (1)-(7) converges uniformly to $\{u^0, \xi^0\}$, that is, to the solution of the problem (1)-(7) for $v = v^0$.

N o t e 3. When the exact solution is nonunique (for example, in Statement II), the quasisolution V_R^* for $R \geq R^0 = \inf_{v^0 \in V^0} \| v^0 \|_{W_2^2 [0, T]}$ coincides with the intersection $V_R \cap V^0$, where $V^0 = \{v^0 \in V, J_g(v^0) = 0\}$ is the set of exact solutions. If $0 < R < R^0$, any element $v_R \in V_R^*$ converges to a certain exact solution $v^0 \in V_{R^0} \cap V^0$ as $R \to R_0$.

N o t e 4. In the general case, without assuming the existence of an exact solution of Eq. (9) for the given right side $g \in G$, we consider a generalized quasisolution on the compact set V_R [8].

4°. One of the key problems in applying the method proposed is to find the gradient of the minimized functional. Work [9] considered the conditions of differentiability of the functionals determined in solutions of the quasilinear Stefan problem, and obtained means of representing the differentials that are convenient for effectively predicting the gradient. We now give an explicit differential expression for the functional $J_g(v) = \| Sv - g \|^2_{L_2[0, T]}$ for Statement I of the boundary-value inverse Stefan problem:

$$dJ_g(v) = \int_0^T a(x, t, u) \psi_x|_{x=0} \Delta v(t)\, dt, \quad v, \Delta v \in V, \tag{17}$$

where the functions $\{\psi(x, t), \kappa(t)\}$ are a solution of the conjugate problem representing a system defined by the relations

$$c(x, t, u) \psi_t = (a(x, t, u) \psi_x)_x + (b(x, t, u) - a_u(x, t, u) u_x) \psi_x +$$

$$+ (b_x(x, t, u) + c_t(x, t, u) - d(x, t) + f_u(x, t, u)) \psi = 0,$$

$$0 < x < \xi(t), \quad \xi(t) < x < l, \quad 0 \leq t < T, \tag{18}$$

$$\psi|_{x=0} = 0, \quad 0 \leq t < T, \tag{19}$$

$$\psi|_{x=\xi(t)} = \kappa(t), \quad 0 \leq t < T, \tag{20}$$

$$a(x, t, u) \psi_x + (b(x, t, u) + e(t)) \psi|_{x=l} = 2(u|_{x=l} - g(t)), \quad 0 \leq t < T, \tag{21}$$

$$\psi|_{t=T} = 0, \quad 0 \leq x \leq l, \tag{22}$$

$$\gamma(x, t, u)|_{x=\xi(t)} \kappa_t + A(x, t, u)|_{x=\xi(t)} \kappa(t) =$$

$$= [a(x, t, u) u_x \psi_x]_{x=\xi(t)}, \quad 0 \leq t < T, \tag{23}$$

$$\kappa|_{t=T} = 0. \tag{24}$$

Here

$$A(x, t, u)|_{x=\xi(t)} = (\gamma_t(x, t, u) + \gamma_u(x, t, u) u_t|_{x=\xi(t)} -$$

$$- \xi_t (\gamma_x(x, t, u) + \gamma_u(x, t, u) u_x)|_{x=\xi(t)} + (\chi_x(x, t, u) +$$

$$+ \chi_u(x, t, u) u_x)|_{x=\xi(t)} + [(a(x, t, u) u_x)_x]_{x=\xi(t)} -$$

$$- [(b(x, t, u) - c(x, t, u) \xi_t) u_x]_{x=\xi(t)};$$

$\{u, \xi\}$ is the solution of the direct Stefan problem (1)-(7) corresponding to the boundary function $v \in V$.

Specifying the form of the minimized functional for various statements of inverse Stefan problems and examining the corresponding initial and boundary conditions for Eqs. (18) and (23), we may derive a differential expression for each of the statements. In the case of the uniphase Stefan problem, Eq. (23) will involve, instead of steps of the functions at $x = \xi(t)$, values of these functions at $x = \xi(t)$.

The devised method of representing the differentials permits construction of effective numerical algorithms for finding quasisolutions of the inverse Stefan problems [4, 5, 7, 8].

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